

3.1: Introduction to Second-Order Linear Equations

A second-order linear differential equation is of the form

$$\frac{d^2y}{dx^2} = f(x, y, \frac{dy}{dx}) \quad \text{or} \quad y'' = f(x, y, y'). \quad (1)$$

(1) can be rearranged as

$$G(x, y, y', y'') = 0 \quad (2)$$

or

$$A(x)y'' + B(x)y' + C(x)y = F(x) \quad (3)$$

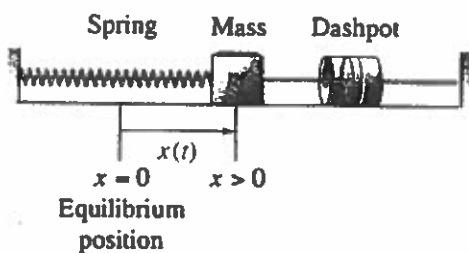
or

$$y'' + p(x)y' + q(x)y = f(x). \quad (4)$$

Definition 1.

- (a) Anytime that the right-hand side of (3) is identically 0; i.e. $F(x) = 0$, the second-order linear equation is called **homogeneous**.
- (b) If the differential equation is not homogeneous it is called **nonhomogeneous**.

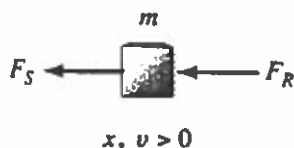
A Typical Application. Linear differential equations frequently appear as mathematical models of mechanical systems and electrical circuits.



Using Newton's Second Law of Motion ($F = ma$) along with the diagrams to the left we get the second-order linear differential equation

$$mx'' = F_S + F_R \quad \text{or} \quad mx'' + cx' + kx = F(t). \quad (5)$$

When $F(t) = 0$ in (5) we get a homogeneous equation which describes the free vibrations of the mass. When $F(t) \neq 0$ in (5) we get a nonhomogeneous equation which describes the forced vibrations of the mass under the influence of an external force $F(t)$.



Theorem 1. (Principle of Superposition for Homogeneous Solutions)

Let y_1 and y_2 be two solutions to the homogeneous linear equation (4); i.e. $f(x) = 0$. If c_1 and c_2 are constants, then the linear combination

$$y = c_1 y_1 + c_2 y_2$$

is also a solution to (4).

Example 1. First check that $y_1 = \cos x$ and $y_2 = \sin x$ are solutions to the differential equation

$$y'' + y = 0.$$

Then check that $y = c_1 y_1 + c_2 y_2$ is also a solution for any $c_1, c_2 \in \mathbb{R}$.

$$\left. \begin{array}{l} y_1 = \cos x \\ y_1' = -\sin x \\ y_1'' = -\cos x \end{array} \right\} -\cos x + \cos x = 0 \checkmark \quad \left. \begin{array}{l} y_2 = \sin x \\ y_2' = \cos x \\ y_2'' = -\sin x \end{array} \right\} -\sin x + \sin x = 0 \checkmark$$

$$\begin{aligned} y &= c_1 \cos x + c_2 \sin x \\ y' &= -c_1 \sin x + c_2 \cos x \\ y'' &= -c_1 \cos x - c_2 \sin x \end{aligned} \checkmark$$

Theorem 2. (Existence and Uniqueness for Linear Equations)

Suppose that p , q and f are continuous on the open interval I in (4) with I containing the point a . Then, given any numbers b_0 and b_1 , (4) has a unique solution with the initial conditions

$$y(a) = b_0 \quad y'(a) = b_1.$$

Example 2. We saw in Example 1 that $y = b_0 \cos x + b_1 \sin x$ is a solution to

$$y'' + y = 0.$$

Solve for b_0 and b_1 given that $y(0) = 3$ and $y'(0) = -2$.

$$\left. \begin{array}{l} (-b_0 \cos x - b_1 \sin x) + (b_0 \cos x + b_1 \sin x) = 0 \\ y(0) = 3 \Rightarrow b_0 = 3, \quad y'(0) = -2 \Rightarrow b_1 = -2. \end{array} \right\} y = 3 \cos x - 2 \sin x$$

Exercise 1. Verify that $y_1 = e^x$ and $y_2 = x e^x$ are solutions to the differential equation

$$y'' - 2y' + y = 0.$$

Next find the unique solution with the initial conditions $y(0) = 3$ and $y'(0) = 1$.

$$\left. \begin{array}{l} y_1 = e^x \\ y_1' = e^x \\ y_1'' = e^x \end{array} \right\} e^x - 2e^x + e^x = 0 \checkmark$$

$$\left. \begin{array}{l} y_2 = x e^x \\ y_2' = e^x + x e^x \\ y_2'' = e^x + e^x + x e^x \end{array} \right\} (2e^x + x e^x) - 2(e^x + x e^x) + x e^x = 0 \checkmark$$

$$y = c_1 e^x + c_2 x e^x. \quad y(0) = 3 \Rightarrow c_1 = 3, \quad y'(0) = 1 \Rightarrow c_2 = -2.$$

Definition 1. Two functions defined on an open interval I are said to be **linearly independent** if neither is a constant multiple of the other. If they are not linearly independent, they are said to be **linearly dependent**.

General Solutions. Consider (4) when $p(x)$ and $q(x)$ are constant and $f(x) = 0$. The resulting homogeneous differential equation is given by

$$y'' + py' + qy = 0. \quad (6)$$

Theorem 2 tells us that in this case there are always two linearly independent solutions.

Theorem 3. (Wronskians of Solutions)

Given two solutions to the homogeneous equation

$$y'' + p(x)y' + q(x)y = 0,$$

the **Wronskian** is defined as

$$W = \det \begin{bmatrix} f & g \\ f' & g' \end{bmatrix} = fg' - gf'.$$

(a) If f and g are linearly independent, then $W(f, g) \equiv 0$.

(b) If f and g are linearly dependent, then $W(f, g) \neq 0$ at each point.

Theorem 4. (General Solutions of Homogeneous Equations)

If y_1 and y_2 are linearly independent solutions to the homogeneous equation

$$y'' + p(x)y' + q(x)y = 0,$$

and p and q are continuous on I , then every solution y is of the form

$$y(x) = c_1y_1(x) + c_2y_2(x)$$

for some constants c_1 and c_2 .

Example 3. Verify that $y_1 = e^{2x}$ and $y_2 = e^{-2x}$ are solutions to

$$y'' - 4y = 0.$$

Next verify that $y_3 = \cosh 2x$ and $y_4 = \sinh 4x$ are also solutions. What does this imply about y_3 and y_4 with respect to y_1 and y_2 ?

Easy to check. It implies that $\cosh 2x$ and $\sinh 2x$ are both linearly dependent w/ e^{2x} and e^{-2x} . You can check that $\cosh 2x = \frac{1}{2}e^{2x} + \frac{1}{2}e^{-2x}$ and $\sinh 2x = \frac{1}{2}e^{2x} - \frac{1}{2}e^{-2x}$.

Linear Second-Order Equation with Constant Coefficients.

We return our attention to (6). We replace $y^{(n)}$ with r^n ; i.e. $y = 1$, $y' = r$, $y'' = r^2$, etc. to get the **characteristic equation** for (6)

$$r^2 + pr + q = 0. \quad (7)$$

Theorem 5. (Distinct Real Roots)

If the roots r_1 and r_2 of the characteristic equation (7) are real and distinct, then

$$y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$$

is a general solution to (6).

Theorem 6. (Repeated Roots)

If the characteristic equation (7) has repeated (necessarily real) roots $r_1 = r_2$, then

$$y(x) = (c_1 + c_2 x) e^{r_1 x}$$

is a general solution to (6).

Example 4. Find the general solution of

$$2y'' - 7y' + 3y = 0.$$

$$\left. \begin{array}{l} 2r^2 - 7r + 3 = 0 \\ (2r-1)(r-3) = 0 \\ r_1 = \frac{1}{2}, r_2 = 3 \end{array} \right\} \text{Thus } y(x) = c_1 e^{x/2} + c_2 e^{3x}$$

Example 5. Solve the initial value problem

$$y'' + 2y' + y = 0, \quad y(0) = 5, y'(0) = -3.$$

$$\left. \begin{array}{l} r^2 + 2r + 1 = 0 \\ (r+1)^2 = 0 \\ r_1 = r_2 = -1 \end{array} \right\} \text{Thus } y(x) = (c_1 + c_2 x) e^{-x} \left. \begin{array}{l} y(0) = 5 \Rightarrow c_1 = 5 \\ y'(0) = -3 \Rightarrow c_2 = 2 \end{array} \right\} \text{Thus } y(x) = (5 + 2x) e^{-x}.$$

Homework. 1-47 (odd)